TORSION OF THIN-WALLED RODS WITH CLOSED CROSS-SECTIONS SUBJECT TO TRANSIENT CREEP

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PMM Vol.22, No.6, 1958, pp.766-780

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(Received 7 July 1958)

This paper deals with the problem of torsion of thin-walled rods of closed cross-section subject to transient creep and variation of the modulus of instantaneous deformation of the material. In its linear formulation, this problem has been studied in the paper [1] by Arutiunian and Chobanian. At the same time, experimental investigations have shown that for a wide stress range ($1/2\,R \leqslant \sigma \leqslant R$, where R is the ultimate strength of the material) the linear relation between creep strains and the corresponding stresses is violated.

The solution of the problem under consideration will be based on the nonlinear equations of creep theory developed in [2], which are known to apply for a wide range of variation of stresses $0 \leqslant \sigma \leqslant R$ for materials such as concrete [3], wood [4], solid soil [5], phenolic plastics [6], etc.

In view of the thin-walled nature of the rods, it will be assumed that the shear stresses are constant through the thickness of the wall of the section and that they act parallel to its middle surface.

This paper gives a generalization of Bredt's theorem on the circulation of shear stress in the case of torsion of prismatic rods of arbitrary cross-section for the case of transient creep and varying modulus of instantaneous deformation. Further, using this theorem, it solves the problem of transient creep in a thin-walled rod with a closed cross-section for torsion by a constant moment M. It also considers the relaxation problem of torsion of such a thin-walled rod subject to transient creep obeying an arbitrary nonlinear law. The solution of this problem is reduced to the solution of a nonlinear Volterra integral equation of the second kind, a study and method of solution of which is given in Section 2.

As an application of this method, Section 3 presents the solution of the problem of relaxation of a twisting moment in a thin-walled tube for a power law between the creep strains and stresses.

1. Connection between strains and stresses for nonlinear creep. In the general case of a three-dimensional state of stress, the equations,* relating the strain intensity $\epsilon_i(t)$ and the stress intensity $\sigma_i(t)$ in the presence of creep of the material and variations of its modulus of instantaneous deformation, will have the following form [2],

$$\varepsilon_i(t) = \frac{\sigma_i(t)}{3G(t)} - \int_{\tau_1}^{t} \sigma_i(\tau) \frac{\partial}{\partial \tau}\left[\frac{1}{3G(\tau)}\right] d\tau - \int_{\tau_1}^{t} F[\sigma_i(\tau)]\, \sigma_i(\tau) \frac{\partial C(t,\tau)}{\partial \tau}\, d\tau \qquad (1.1)$$

Here $C(t, \tau)$ is the measure of the creep of the material for uniaxial stress, $F(\sigma_i)$ is a certain function characterizing the nonlinear dependence between the creep stresses and strains for a given material which has been determined experimentally by tests on simple creep specimen and normalized to the condition $F(1) = 1$, $G(t)$ is the modulus of instantaneous deformation in shear, τ_1 is the age of the material at the moment when the load is applied, and t is the time.

Under these conditions we have

$$\varepsilon_i(t) = \frac{\sqrt{2}}{3} \sqrt{(\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2 + \tfrac{3}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2)} \qquad (1.2)$$

$$\sigma_i(t) = \frac{\sqrt{2}}{2} \sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2)} \qquad (1.3)$$

Using the usual transformation formulas, relating the stress and strain components in a rectangular system of coordinates to the corresponding components in the system of principal axes and taking into consideration that the stress and strain deviators have identical principal directions at any instant of time t, from (1.1) we get

$$\varepsilon_x(t) = \frac{1}{2G(t)}\left[\sigma_x(t) - \frac{3\nu}{1+\nu}\sigma(t)\right] - \int_{\tau_1}^{t}\left[\sigma_x(\tau) - \frac{3\nu}{1+\nu}\sigma(\tau)\right]\frac{\partial}{\partial\tau}\left[\frac{1}{2G(\tau)}\right] d\tau -$$

$$- \int_{\tau_1}^{t} [\sigma_x(\tau) - \sigma(\tau)]\, F[\sigma_i(\tau)] \frac{\partial C^*(t,\tau)}{\partial \tau}\, d\tau \qquad (1.4)$$

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* For the sake of brevity, x, y, z designations are omitted.

$$\gamma_{xy}(t) = \frac{\tau_{xy}(t}{G(t)} - \int_{\tau_1}^{t} \tau_{xy}(\tau) \frac{\partial}{[\partial\tau} \left[\frac{1}{G(\tau)}\right] d\tau - 2\int_{\tau_1}^{t} \tau_{xy}(\tau) F[\sigma_i(\tau)] \frac{\partial C^*(t, \tau)}{\partial\tau} d\tau \quad (x, y, z)$$

where ν is the coefficient of transverse contraction of the material for the elastic part of the strain

$$\sigma(t) = \tfrac{1}{3}[\sigma_x(t) + \sigma_y(t) + \sigma_z(t)], \quad C^*(t, \tau) = \tfrac{3}{2} C(t, \tau)$$

Here and in what follows, the symbol (x, y, z) indicates that the remaining relations are to be obtained by cyclic rearrangement of x, y, z.

Thus, the relations (1.4) relate the strain and stress components in the case of a three-dimensional state of stress, including creep of the material, and variations of its modulus of instantaneous deformation. They have been deduced on the basis of the assumption that the volume of the body changes elastically, i.e. that there are no volume changes due to creep, a circumstance which has been confirmed by numerous experimental studies. It has of course been assumed for this purpose that the coefficient of transverse contraction in creep is $\mu = 1/2$ and that the measure of creep for pure shear $\omega(t, \tau)$ is related to $C(t, \tau)$ by

$$\omega(t, \tau) = 2[1 + \mu(t, \tau)] C(t, \tau) = 3C(t, \tau) \tag{1.5}$$

It will be noted that the equations (1.4) describe a process of deformation which includes ageing as well as heredity of the material, and which also applies to the case of active strain when the nonlinearity depends only on time.

To describe processes of deformation without ageing of the material, the nonlinear equations of the theory of creep and relaxation were given by Rabotnov [7] and Rozovskii [8].

For the sole purpose of simplifying the following calculations it will be assumed that

$$G(t) = G = \mathrm{const}$$

Then equation (1.4) takes the form

$$\varepsilon_x(t) = \frac{1}{2G}\left[\sigma_x(t) - \frac{3\nu}{1+\nu}\sigma(t)\right] - \int_{\tau_1}^{t} [\sigma_x(\tau) - \sigma(\tau)] F[\sigma_i(\tau)] \frac{\partial C^*(t, \tau)}{\partial\tau} d\tau$$

$$\gamma_{xy}(t) = \frac{\tau_{xy}(t)}{G} - 2\int_{\tau_1}^{t} \tau_{xy}(\tau) F[\sigma_i(\tau)] \frac{\partial C^*(t, \tau)}{\partial\tau} d\tau \tag{1.6}$$

Let $F(\sigma\ )$ have the form

$$F(\sigma_i) = \alpha + \beta f^*(\sigma_i) \tag{1.7}$$

where α and β are constant parameters and $f^*(\sigma_i)$ satisfies the conditions

$$\alpha + \beta f^*(1) = 1, \qquad f^*(\sigma_i) > 0, \qquad f^{*\prime}(\sigma_i) > 0 \tag{1.8}$$

Thus the problem will be considered for an arbitrary nonlinear law.

Obviously, if the parameter β in (1.7) is small, the function $F(\sigma_i)$ will describe the creep curve of a weakly nonlinear material.

Substituting for $F(\sigma_i)$ from (1.7) in (1.6), one finds

$$\varepsilon_x(t) = \frac{1}{2G}\left[\sigma_x(t) - \frac{3\nu}{1+\nu}\sigma(t)\right] - 3\alpha\int_{\tau_1}^{t}[\sigma_x(\tau) - \sigma(\tau)]\frac{\partial C\ (t,\tau)}{\partial\tau}d\tau -$$

$$-3\beta\int_{\tau}^{t}[\sigma_x(\tau) - \sigma(\tau)]\, f^*[\sigma_i(\tau)]\frac{\partial C\_(t,\tau)}{\partial\tau}d\tau \tag{1.9}$$

$$\gamma_{xy}(t) = \frac{\tau_{xy}(t)}{G} - 3\alpha\int_{\tau_1}^{t}\tau_{xy}(\tau)\frac{\partial C(t,\tau)}{\partial\tau}d\tau - 3\beta\int_{\tau_1}^{t}\tau_{xy}(\tau)\, f^*[\sigma_i(\tau)]\frac{\partial C(t,\tau)}{\partial\tau}d\tau$$

It should be noted that, as has been shown by experimental studies [3,4], actual creep curves for a number of materials (e.g. concrete, wood, etc.) are in many cases approximated sufficiently well for high stresses ($\sigma \geqslant 1/2\ R$) by power laws. In this case in (1.9) one may put

$$f^*(\sigma_i) = \sigma_i^{\,m} \qquad (m > 0) \tag{1.10}$$

2. The generalized law of Bredt. Consider a prismatic rod with arbitrary cross-section whose material is subject to creep and has a constant modulus of instantaneous deformation. Let the side surfaces of the rod be free from external forces and let forces which induce a constant torque M about the axis of the rod be applied to its ends.

Place the origin of the rectilinear coordinate system x, y, z at some point of an end section of the rod with the Oz axis parallel to the generators.

As in the case of torsion of elastic rods, let all stress and strain components except τ_{xz}, τ_{yz} and γ_{xz}, γ_{yz} vanish at any instant t, i.e. let

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0, \qquad \varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = 0 \tag{2.1}$$

and

$$\gamma_{xz}(t) = \frac{\partial w}{\partial x} - \theta(t)\, y, \qquad \gamma_{yz}(t) = \frac{\partial w}{\partial y} + \theta(t)\, x \tag{2.2}$$

where $\theta(t)$ is the angle of twist per unit length of the rod at time t, $w(t)$

is the displacement component along the axis of the twisted rod.

Then equations (1.9), relating the strain components (2.2) to the corresponding stress components, in the presence of nonlinear creep take the form:

$$\gamma_{xz}(t) = \frac{\tau_{xz}(t)}{G} - 3\alpha \int_{\tau_1}^{t} \tau_{xz}(\tau) \frac{\partial C(t,\tau)}{\partial\tau} d\tau - 3\beta \int_{\tau_1}^{t} \tau_{xz}(\tau) f^*[\sigma_i(\tau)] \frac{\partial C(t,\tau)}{\partial\tau} d\tau \quad (2.3)$$

$$\gamma_{yz}(t) = \frac{\tau_{yz}(t)}{G} - 3\alpha \int_{\tau_1}^{t} \tau_{yz}(\tau) \frac{\partial C(t,\tau)}{\partial\tau} d\tau - 3\beta \int_{\tau_1}^{t} \tau_{yz}(\tau) f^*[\sigma_i(\tau)] \frac{\partial C(t,\tau)}{\partial\tau} d\tau$$

where

$$\sigma_i(t) = \sqrt{3}\sqrt{\tau_{xz}^2(t) + \tau_{yz}^2(t)} \quad (2.4)$$

The equilibrium equations will be identically satisfied if, as is usual, we introduce

$$\tau_{xz} = \frac{\partial\psi}{\partial y}, \qquad \tau_{yz} = -\frac{\partial\psi}{\partial x} \quad (2.5)$$

The stress function ψ depends here only on x, y, t and on the boundary (Γ) of the cross-section of the rod satisfies the condition

$$\left.\frac{\partial\psi}{\partial s}\right|_{\Gamma} = 0 \qquad \text{for } t \geqslant \tau_1 \quad (2.6)$$

since the side faces are free from external forces.

Let L be a closed curve lying entirely in the cross section* of the twisted rod. The curvilinear integral

$$J^* = \oint_L (\gamma_{xz}dx - \gamma_{yz}dy) \quad (2.7)$$

will be called the shear circulation for the closed curve L.

In (2.7) substituting the expressions for the strain components (2.2), and using the condition of single-valuedness of the displacement $w(t)$, we obtain

$$J^* = 2\theta(t)\,\Omega \quad (2.8)$$

where Ω is the area of the region contained in L.

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* In what follows it will be sufficient to require that the boundary of the cross-section of the contour, which in the general case may be multiple-connected, is sectionally smooth.

On the other hand, introducing the expressions for the strain components (2.3) and (2.7), and using (2.5), we find

$$J^* = -\oint_L \left\{ \frac{1}{G} \frac{\partial \psi(t)}{\partial n} - 3\alpha \int_{\tau_1}^{t} \frac{\partial \psi(\tau)}{\partial n} \frac{\partial C(t, \tau)}{\partial \tau} d\tau - \right.$$

$$\left. - 3\beta \int_{\tau_1}^{t} f[\sigma_i(\tau)] \frac{\partial \psi(\tau)}{\partial n} \frac{\partial C(t, \tau)}{\partial \tau} d\tau \right\} ds \tag{2.9}$$

Thus, by (2.8) and (2.9), we must have

$$\oint_L \left\{ \frac{\partial \psi(t)}{\partial n} - 3G \int_{\tau_1}^{t} \left[\alpha + \beta f^*(\sigma_i(\tau))\right] \frac{\partial \psi(\tau)}{\partial n} \frac{\partial C(t, \tau)}{\partial \tau} d\tau \right\} ds = -2\theta(t) G\Omega \tag{2.10}$$

If the instantaneous modulus of shear deformation depends on time, in an analogous manner, using (1.4) and (2.7), we obtain

$$\oint_L \left\{ \frac{\partial \psi(t)}{\partial n} - 3G \int_{\tau_1}^{t} \left[\alpha + \beta f^*(\sigma_i(\tau)) \frac{\partial C(t, \tau)}{\partial \tau} + \frac{\partial}{\partial \tau}\left(\frac{1}{3G(\tau)}\right)\right] \frac{\partial \psi(\tau)}{\partial n} d\tau \right\} ds =$$

$$= -2\theta(t) G\Omega \tag{2.11}$$

For $G(t) = G$ and $C(t, \tau) = 0$, i.e. in the absence of creep and for constant instantaneous modulus of deformation, the relation (2.11) leads to Bredt's well-known formula for elastic rods.

The integral relations (2.10) or (2.11) are generalizations of Bredt's theorem on the circulation of shear stresses during the torsion of prismatic rods in a transient state of creep for an arbitrary nonlinear relationship between the creep strains and stresses and a variable modulus of instantaneous deformation. This theorem, on the basis of its deduction, is the necessary and sufficient condition for the unique determination of the displacements in a twisted rod at any instant t with the help of the stress function $\psi(x, y, t)$.

3. Transient creep and relaxation in torsion of thin-walled rods with closed contours.

1. Transient creep for constant torque. Consider a thin-walled rod whose cross-section is bounded by two closed contours Γ_0 and Γ_1. To simplify the ensuing calculations, the thickness of the wall of the section h will be assumed to be constant. Let the cross-section of the rod be referred to coordinates s, n, where s is measured along the middle line of the contour Γ and n is the normal to s. (Fig. 1).

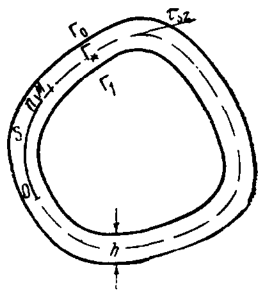

Fig. 1.

In view of the fact that the rod has thin walls, let

$$\tau_{sz}(t) = -\frac{\partial\psi(t)}{\partial n}, \qquad \tau_{nz} = 0 \tag{3.1}$$

At the same time, from the equilibrium conditions we get

$$T = \tau_{sz}(t)\,h = \text{const} \tag{3.2}$$

i.e. the flow of shear stress along the contour of the section at any time t is constant.

It will be noted that the magnitude of the error associated with the assumption (3.1) is in the case of elastic rods of the order $1 + h/R_0 \approx 1$, where h is the thickness of the wall of the cross-section, and R_0 is a typical dimension of the section which is equal to the smaller of the following two quantities, the radius of curvature ρ of the middle line of the profile and its length l. By (3.1) and (3.2), one has

$$\psi(t) = \frac{\psi_1(t)}{2}\left(1 - \frac{2n}{h}\right) \tag{3.3}$$

where $\psi_1(t)$ is the value of the stress function $\psi(t)$ at the inner contour Γ_1 at time t, while at the outer contour Γ_0 one has $\psi(t) = \psi_0(t) = 0$.

By (2.4) and (3.3), one now finds

$$\frac{\partial\psi(t)}{\partial n} = -\frac{\psi_1(t)}{h}, \qquad \sigma_i(t) = \frac{\psi_1(t)}{h} \tag{3.4}$$

$$M(t) = 2\psi_1(t)\,\Omega_* \tag{3.5}$$

where Ω_* is the area of the region bounded by the middle line Γ_*. If the torque M is constant, then

$$\psi_1 = \frac{M}{2\Omega_*} = \text{const} \tag{3.6}$$

Using Bredt's generalized formula (2.10) and the relations (3.4) to (3.6), for the angle of twist $\theta(t)$ we obtain

$$\theta(t) = \frac{Ml}{4h\Omega_*^2}\left\{\frac{1}{G} + 3\left[\alpha + \beta f^*\left(\frac{M}{2h\Omega_*}\right)\right]\right\} C(t, \tau_1) \tag{3.7}$$

and, by (2.2) and (2.4), for the shear stress

$$\tau_{sz} = \frac{M}{2h\Omega^*} \tag{3.8}$$

which coincides with the values of the shear stress corresponding to the elastic case.

Thus it follows from this solution that the creep of the material only influences the deformation of the twisted rod, while the stress deformation remains the same as in the elastic, instantaneous case.

On the other hand, it follows from the general equations of the theory of creep [2] that under conditions of linear creep the stresses in a twisted rod do not vary with time and that they coincide with the values of the stresses corresponding to the elastic instantaneous problem, while for nonlinear creep the stresses change with time and differ from the elastic stresses occurring in the rod at instant t when the twisting moments are applied to its ends.

It appears that the contradiction between the last two deductions is explained by the fact that the first conclusion is approximate inasmuch as it assumes the character of the distribution of the shear stresses in the thin-walled rod.

In other words, the stresses in the above case are elastic only within a small quantity of order h/R_0.

As regards the strains, in the case of nonlinear creep, they are determined by (3.7) exactly, apart from small terms.

2. Relaxation problem. At a certain instant $t = \tau_1$, let a thin-walled rod with closed section be twisted by the moment $M(\tau_1)$, and then let its ends be clamped. As we know, as a consequence of creep, the stresses in the clamped rod will in time subside.

The relaxation problem of torsion consists of the determination of the law of variation of the stresses or the torque in time as a function of the creep properties of the material and the initial twist of the rod:

$$\theta(\tau_1) = \frac{M(\tau_1)\, l}{4Gh\Omega_*^2}$$

Using Bredt's generalized formula (2.10) and the relations (3.4), (3.6), (3.8) for the determination of $\tau_{sz}(t)$, we obtain the following integral equation

$$\tau_{sz}(t) - \alpha \int_{\tau_1}^{t} \tau_{sz}(\tau) K(t,\tau)\, d\tau - \beta \int_{\tau_1}^{t} f^*(\tau_{sz}(\tau)) K(t,\tau) \tau_{sz}(\tau)\, d\tau = g(\tau_1) \quad (3.9)$$

where

$$K(t,\tau) = 3G \frac{\partial C(t,\tau)}{\partial \tau}, \qquad g(\tau_1) = \frac{2\theta(\tau_1) G\Omega_*}{l} = \frac{M(\tau_1)}{2h\Omega_*} \quad (3.10)$$

Thus, the solution of the problem of relaxation of stresses in the torsion of thin-walled prismatic rods of closed section, in the presence of transient creep obeying an arbitrary nonlinear law, has been reduced to the solution of the nonlinear Volterra integral equation of the second kind (3.9).

Having found the stress $\tau_{sz}(t)$ from (3.9), the relaxation of the torque $M(t)$ is determined by (3.8).

In conclusion, it will be noted that, if the twist θ of the rod is not constant but is instead a given function of time $\theta = \theta(t)$, then the relaxation problem in so general a formulation leads to the solution of the same integral equation (3.9), the only difference being that its right-hand side will no longer be constant but a given function of time. A method of solution and a study of the equation (3.9) for this general case is given in the next section.

4. Method of solution of the basic integral equation and its study. This section gives the solution and study of the basic nonlinear integral equation (3.9), which may for convenience be written in the form:

$$u(t) - \alpha \int_0^t K(t,\tau) u(\tau)\, d\tau - \beta \int_0^t K(t,\tau) f[u(\tau)]\, d\tau = g(t) \quad (4.1)$$

where

$$\tau_{sz}(t) = u(t), \qquad K(t,\tau) = 3G \frac{\partial C(t,\tau)}{\partial \tau}$$
$$f(u) = u f^*(u), \qquad g(t) = \frac{2\theta(t) G\Omega_*}{l} \quad (4.2)$$

In the integral equation (4.1), $0 < t, \tau < +\infty$, $g(t)$ and $K(t, \tau)$ are continuous functions, α, β are numerical parameters and $f(u)$ is an analytic function (a bound on which will be established below).

Let $R(t, \tau, \alpha)$ be the resolvent of the linear Volterra integral equation with Kernel $K(t, \tau)$, when the equation (4.1) may be replaced by an equivalent equation of the form

$$u(t) = h(t) + \beta \int_0^t H(t, \tau) f[u(\tau)]\, d\tau \tag{4.3}$$

Here

$$h(t) = g(t) + \alpha \int_0^t R(t, \tau, \alpha)\, g(\tau)\, d\tau$$

$$H(t, \tau) = K(t, \tau) + \int_\tau^t R(t, \xi, \alpha)\, K(\xi, \tau)\, d\xi \tag{4.4}$$

The solution of (4.3)will be assumed to be analytic in β, and an attempt will be made to solve it by expanding it in a Taylor series in powers of β. Let

$$u(t, \beta) = \sum_{k=0}^{\infty} \frac{v_k(t)}{k!} \beta^k \tag{4.5}$$

$$\Phi(t, \beta) = f[u(t, \beta)] = \sum_{k=0}^{\infty} \frac{\Phi_k(t)}{k!} \beta^k \tag{4.6}$$

where obviously

$$v_k(t) = \left.\frac{\partial^k u(t, \beta)}{\partial \beta^k}\right|_{\beta=0}, \qquad \Phi_k(t) = \left.\frac{\partial^k \Phi(t, \beta)}{\partial \beta^k}\right|_{\beta=0} \tag{4.7}$$

Substituting (4.5) in (4.3), taking (4.6) into consideration and comparing coefficients of the same powers of β, one finds

$$v_{n+1}(t) = (n + 1) \int_0^t H(t, \tau)\, \Phi_n(\tau)\, d\tau, \qquad v_0(t) = h(t) \tag{4.8}$$

Using a known formula for the n-th derivative of a complex function, it may be proved that

$$\Phi_n(t) = n! \sum_n \frac{1}{i_1! i_2! \ldots i_n!} \left(\frac{v_1(t)}{1!}\right)^{i_1} \left(\frac{v_2(t)}{2!}\right)^{i_2} \ldots \left(\frac{v_n(t)}{n!}\right)^{i_n} f^{(i_1+i_2+\ldots+i_n)}(v_0(t)) \tag{4.9}$$

where the sum extends over all integral nonnegative solutions of the equation $i_1 + 2i_2 + 3i_3 + \ldots + ni_n = n$.

Substituting (4.9) into (4.8), we obtain

$$\frac{v_{n+1}}{(n+1)!} = \int_0^t H(t,\tau) \sum_n \frac{1}{i_1! i_2! \ldots i_n!} \left(\frac{v_1(\tau)}{1!}\right)^{i_1} \ldots \left(\frac{v_n(\tau)}{n!}\right)^{i_n} f^{(i_1+i_2+\ldots+i_n)}(h(\tau))\, d\tau$$
$$(n = i_1 + 2i_2 + \ldots + ni_n)$$

or

$$\frac{v_{n+1}}{(n+1)!} = \sum_n \frac{1}{i_1! i_2! \ldots i_n!} \int_0^t H(t,\tau) \left(\frac{v_1(\tau)}{1!}\right)^{i_n} \ldots \left(\frac{v_n(\tau)}{n!}\right)^{i_n} f^{(i_1+i_2+\ldots+i_n)}(h(\tau))\, d\tau$$
$$(n = i_1 + 2i_2 + \ldots + ni_n) \tag{4.10}$$

Formula (4.10) offers the possibility of determining successively all $v_n(t)$, whence in particular follows the uniqueness of the solution, analytic in β, of the equation (4.3) or (4.1), provided such a solution exists.

In order to prove the existence of such a solution, we must obtain an estimate of $|v_n(t)|$ which depends on n uniformly with respect to t in some interval. Let there be given some fixed interval of time $0 \leqslant t \leqslant T$. With

$$\max|K(t,\tau)| = K_T, \quad \max|R(t,\tau,\alpha)| = R_T, \quad H_T = K_T R_T \tag{4.11}$$
$$(0 \leqslant t,\ \tau \leqslant T)$$

$$\inf h(t) = a_T, \quad \sup h(t) = b_T \quad (0 \leqslant t \leqslant T) \tag{4.12}$$

we obviously have

$$|H(t,\tau)| \leqslant H_T |t-\tau| \tag{4.13}$$

It will now be assumed that $f(u)$ is determined in $a_T \leqslant u \leqslant b_T$ and it has three derivatives of all orders which satisfy the inequalities

$$|f^{(p)}(u)| \leqslant K_0 p! a^p \qquad (p = 0, 1, 2, \ldots) \tag{4.14}$$

where a and K_0 are constants. Then from the recurrence relations (4.10), taking into consideration (4.14), we get:

$$|v_{n+1}(t)| \leqslant (n+1)! \sum_n \frac{(i_1+i_2+\ldots+i_n)!\, H_T K_0 a^{i_1+i_2+\ldots+i_n}}{i_1! i_2! \ldots i_n!} \times$$
$$\times \int_0^t (t-\tau) \left|\frac{v_1(\tau)}{1!}\right|^{i_1} \ldots \left|\frac{v_n(\tau)}{n!}\right|^{i_n} d\tau \qquad (n = i_1 + 2i_2 + \ldots + ni_n) \tag{4.15}$$

Letting $R_0 = aK_0 H_T$, it will now be proved that for all n

$$|v_n(t)| \leqslant \frac{A_n}{a} R_0^n t^{2n} \tag{4.16}$$

where A_n is a constant which only depends on n.

The inequality (4.16) will be proved by mathematical induction. Letting $n = 0$ in (4.10), we obtain

$$v_1(t) = \int_0^t H(t, \tau) f(h(\tau))\, d\tau$$

Hence

$$|v_1(t)| \leqslant K_0 H_T \int_0^t (t-\tau)\, d\tau = \frac{K_0 H_T t^2}{2}$$

or

$$|v_1(t)| \leqslant \frac{A_1}{a} R_0 t^2, \qquad A_1 = \frac{1}{2}$$

i.e. the inequality (4.16) is true for $n = 1$.

It will next be assumed that the inequality is true for $k \leqslant n$, and it will be shown to be then true for $k = n + 1$ likewise. Substituting (4.16) into (4.15), we find

$$|v_{n+1}(t)| \leqslant (n+1)!\, K_0 H_T \sum_n \frac{(i_1 + i_2 + \ldots + i_n)!}{i_1!\, i_2! \ldots i_n!} \int_0^t (t-\tau) R_0^{\,i_1+2i_2+\ldots+ni_n} \times$$

$$\times \tau^{2(i_1+2i_2+\ldots+ni_n)} \left(\frac{A_1}{1!}\right)^{i_1} \ldots \left(\frac{A_n}{n!}\right)^{i_n} d\tau =$$

$$= \frac{(n+1)!}{a} R_0^{n+1} \sum_n \frac{(i_1 + i_2 + \ldots + i_n)!}{(2n+1)(2n+2)\, i_1! \ldots i_n!} \left(\frac{A_1}{1!}\right)^{i_1} \left(\frac{A_2}{2!}\right)^{i_2} \ldots \left(\frac{A_n}{n!}\right)^{i_n} t^{2n+2} =$$

$$= \frac{1}{a} \left\{ \frac{(n+1)!}{(2n+1)(2n+2)} \sum_n \frac{(i_1 + i_2 + \ldots + i_n)!}{i_1!\, i_2! \ldots i_n!} \left(\frac{A_1}{1!}\right)^{i_1} \left(\frac{A_2}{2!}\right)^{i_2} \ldots \left(\frac{A_n}{n!}\right)^{i_n} \right\} R_0^{n+1} t^{2n+2}$$

$$(n = i_1 + 2i_2 + \ldots + ni_n)$$

i.e.

$$|v_{n+1}(t)| \leqslant \frac{A_{n+1}}{a} R_0^{n+1} t^{2(n+1)}$$

where

$$A_{n+1} = \frac{(n+1)!}{(2n+1)(2n+2)} \sum_n \frac{(i_1 + i_2 + \ldots + i_n)!}{i_1!\, i_2! \ldots i_n!} \left(\frac{A_1}{1!}\right)^{i_1} \left(\frac{A_2}{2!}\right)^{i_2} \ldots \left(\frac{A_n}{n!}\right)^{i_n}$$

$$(n = i_1 + 2i_2 + \ldots + ni_n) \tag{4.17}$$

Thus the inequality (4.16) will be true for all $n \geqslant 1$, provided the constants A_n are determined from the recurrence relations (4.17). It will be important for what follows to obtain an estimate of the absolute constants A_n, determined uniquely by (4.17).

For the sake of convenience, let $B_k = A_k/k!$ when (4.17) gives

$$B_{n+1} = \frac{1}{(2n+1)(2n+2)} \sum_{n} \frac{(i_1+i_2+\ldots+i_n)!}{i_1!\, i_2! \ldots i_n!} B_1^{i_1} B_2^{i_2} \ldots B_n^{i_n}$$
$$(n = i_1 + 2i_2 + \ldots + ni_n) \tag{4.18}$$

which is again a recurrence relation, uniquely determining all the constants $B_k (k \geqslant 1)$. Obviously all $B_k > 0$, and if we were to replace in (4.18) B_k by $C_k \geqslant B_k (k = 1, 2, \ldots, n)$, or to increase the coefficients of $B_1^{i_1} B_2^{i_2}, \ldots, B_n^{i_n}$, this would only increase the value of B_{n+1}.

It will now be proved that one may select positive constants A and α_0 such that for all $k \geqslant 1$

$$B_k \leqslant A\alpha_0^k \tag{4.19}$$

Let it be assumed that A and α_0 are such that the inequality (4.19) is fulfilled for all $k \leqslant n$. It then follows from (4.18) that

$$B_{n+1} \leqslant \frac{1}{(2n+1)(2n+2)} \sum_{n} \frac{(i_1+i_2+\ldots+i_n)!}{i_1!\, i_2! \ldots i_n!} A^{i_1+i_2+\ldots+i_n} \alpha_0^{i_1+2i_2+\ldots+ni_n} =$$
$$= \frac{\alpha_0^n}{(2n+1)(2n+2)} \sum_{n}' \sum_{m,n} \frac{(i_1+i_2+\ldots+i_n)!}{i_1!\, i_2! \ldots i_n!} A^{i_1+i_2+\ldots+i_n}$$
$$(n = i_1 + 2i_2 + \ldots + ni_n)$$

or

$$B_{n+1} \leqslant \frac{\alpha_0^n}{(2n+1)(2n+2)} \sum_{m=0}^{n}{}' \sum_{n}{}' \frac{m!}{i_1!\, i_2! \ldots i_n!} A^m \begin{pmatrix} n = i_1 + 2i_2 + \ldots + ni_n \\ m = i_1 + i_2 + \ldots + i_n \end{pmatrix} \tag{4.20}$$

The second sum in (4.20) extends over all nonnegative solutions of the system of equations

$$i_1 + 2i_2 + 3i_3 + \ldots + ni_n = n \tag{4.21}$$
$$i_1 + i_2 + i_3 + \ldots + i_n = m \tag{4.22}$$

The inequality (4.20) will be increased if the equation (4.21) is disregarded, the second summation extended over all nonnegative solutions of the equations (4.22) and the result then divided by the number of all possible rearrangements of n elements, i.e. by $n!$, since the omitted equation made such permutations impossible in the same way as equation (4.22) made them possible. Thus

$$B_{n+1} \leqslant \frac{\alpha_0^n}{(2n+1)(2n+2)} \sum_{m=0}^{n}{}' \frac{A^m}{n!} \sum_{m}{}' \frac{m!}{i_1!\, i_2! \ldots i_n!} = \frac{\alpha_0^n}{(2n+1)(2n+2)n!} \sum_{m=0}^{n}{}' (nA)^m$$

where use has been made of the fact that

$$\sum_m \frac{m!}{i_1! \ldots i_n!} \qquad (m = i_1 + i_2 + \ldots + i_n)$$

is known to represent the sum of the coefficients in the expansion $(a_1 + a_2 + \ldots + a_n)^m$ and to be equal to n^m.

Hence we arrive at the inequality

$$B_{n+1} \leqslant \left\{ \frac{(nA)^{n+1} - 1}{(nA - 1)\,\alpha} \, \frac{1}{n!\,(2n+1)(2n+2)} \right\} \alpha_0^{n+1} \tag{4.23}$$

If we were to find that the selected A and a_0 was such that for all natural n the inequality

$$\frac{(nA)^{n+1} - 1}{\alpha_0 (nA - 1)} \, \frac{1}{n!\,(2n+1)(2n+2)} \leqslant A \tag{4.24}$$

was fulfilled, it would follow from (4.23) that for such A and a_0 the inequality (4.19) was also satisfied for $k = n + 1$, i.e. for all natural n.

It will now be shown that for the inequality (4.24) to hold true, the constants A and a_0 must satisfy the inequalities $eA \leqslant 1$ and $2\,a_0 A \geqslant 1$. In fact, the latter is obtained immediately from (4.24) for $n = 0$.

To prove the inequality $eA \leqslant 1$, use will be made of Stirling's asymptotic formula

$$n! \sim \sqrt{2\pi n}\, e^{-n} n^n$$

from which for large values of n there follows

$$\frac{(nA)^{n+1} - 1}{\alpha_0 (nA - 1)} \, \frac{1}{n!\,(n2+1)(2n+2)} \approx \frac{(nA)^n}{\alpha_0} \, \frac{1}{\sqrt{2\pi n e^{-n} n^n}\,(2n+1)(2n+2)}$$

or

$$\frac{(nA)^{n+1} - 1}{\alpha_0 (nA - 1)} \, \frac{1}{n!\,(2n+1)(2n+2)} \sim \frac{(eA)^n}{\alpha_0 \sqrt{2\pi n}\,(2n+1)(2n+2)} \tag{4.25}$$

It follows from (4.25) that for $Ae > 1$, the inequality (4.24) may not be satisfied for sufficiently large n, while for such n as $Ae \leqslant 1$ this relation is $O(n^{-5/2})$. Obviously the best choice of A and a_0 is $A = e^{-1}$, $a_0 = 1/2\,e$. Finally, it will be proved that for this choice of the constants A and a_0 checked for $n = 0$, and sufficiently large n, the inequality (4.24) holds true for all n without exception. For $n = 1, 2$, the inequality is easily verified direct. It remains to be proved that for all $n \geqslant 3$

$$\frac{(n/e)^{n+1} - 1}{(n+1)!\,(2n+1)(n/e - 1)} \leqslant 1 \tag{4.26}$$

By a known approximation [10] for $n!$ we have

$$n! = \sqrt{2\pi n}\, e^{-n} n^n \left(1 + \frac{\omega}{\sqrt{2\pi n}}\right). \qquad |\omega| < 1$$

Hence we find $(n+1)! > 4\, e^{-(n+1)} (n+1)^{n+1}$ for $n \geqslant 3$, and therefore

$$\frac{(n/e)^{n+1} - 1}{(n+1)!\,(2n+1)\,(n/e-1)} < \frac{n^{n+1} e^{-(n+1)} e}{4e^{-(n+1)}\,(n+1)^{n+1}\,(2n+1)\,(n-e)} < \frac{e}{4 \cdot 7 \cdot 0.2} < 1$$

The inequality (4.26) is thus true for all natural n, and it follows from these inequalities by virtue of the above statements that the inequality (4.24), and therefore also (4.19), hold true for all n, provided we put in $A \leqslant e^{-1}$, $a_0 \geqslant e/2$.

It has now been proved that for all natural n

$$B_n \leqslant \frac{1}{e}\left(\frac{e}{2}\right)^n \tag{4.27}$$

whence, recalling the notation $A_n = n! B_n$, the inequality (4.26) may be written in its final form

$$|v_n(t)| \leqslant \frac{1}{e}\left(\frac{e}{2}\right)^n \frac{n!}{a} R_0^n t^{2n} \tag{4.28}$$

It follows from the estimate (4.28) that the series (4.5) for $0 \leqslant t \leqslant T$ is maximized by a series with positive terms

$$\sum_{k=0}^{\infty} \frac{1}{ae}\left(\frac{eR_0T^2|\beta|}{2}\right)^k \tag{4.29}$$

which obviously converges as a geometric sequence, so long as the parameter β satisfies the inequality

$$|\beta| < \frac{2}{eR_0T^2} \tag{4.30}$$

Thus, if condition (4.27) is satisfied, the series (4.5) converges absolutely and uniformly inside the circle (4.27), uniformly with respect to t in the above stated interval, and represents the solution of the basic nonlinear integral equation (4.1).

In conclusion, it will be noted that from a practical point of view a peculiarity of the above method of solution consists in the fact that for its construction we may take the solution of the corresponding linear creep problem as the first approximation, just as in the known solutions [11, 12, 13] of the problem of nonlinear creep theory by the method of successive approximations, the solution of the corresponding elastic problem may be used as the first approximation. In fact, thanks to this feature, the rate of convergence of the successive approximations is considerably increased.

5. Relaxation of torque in a thin-walled rod of tubular cross-section. Let the thin-walled rod of tubular cross-section (Fig. 1) at some instant $t = \tau_1$ be twisted through the angle

$$\theta(\tau_1) = \frac{M(\tau_1)\,l}{4hG\Omega_*^2};$$

which will remain unchanged in what follows. Let the creep property of the material of the rod be characterized by [2]

$$C(t, \tau) = \varphi(\tau)\left[1 - e^{-\gamma(t-\tau)}\right] \tag{5.1}$$

where $\phi(\tau)$ is some monotonically decreasing function characterizing the creep of the material as a function of its age τ, and γ is a constant parameter. Then, by (4.2), the right-hand side and the kernel of the integral equation (4.1) will be

$$K(t, \tau) = 3G\{\varphi'(\tau) - e^{-\gamma(t-\tau)}[\varphi'(\tau) - \gamma\varphi(\tau)]\}, \qquad g(\tau_1) = \frac{M(\tau_1)}{2h\Omega_*} \tag{5.2}$$

The resolvent $R(t, \tau, \alpha)$ of the linear Volterra integral equation with the kernel $K(t, \tau)$ is determined by [2]

$$R(t, \tau, \alpha) = \gamma - \eta'(\tau) + [\eta'^2(\tau) + \eta''(\tau) - \gamma\eta'(\tau)]\, e^{\eta(\tau)} \int_\tau^t e^{-\eta(x)}\, dx \tag{5.3}$$

where

$$\eta(\tau) = \gamma \int_{\tau_1}^{\tau} [1 + 3G\alpha\varphi(\tau)]\, d\tau \tag{5.4}$$

As has been shown in [2], one may write

$$\varphi(\tau) = \frac{A_1}{\tau} + C_0 \tag{5.5}$$

where A_1 and C_0 are constants, characterizing the change in the amount of creep of the material in its early and late stages respectively.

Further, let the nonlinear dependence between the strains and stresses in (4.1) be expressed by the power law

$$f(\tau_{sz}) = f^*(\tau_{sz})\,\tau_{sz} = \tau_{sz}^{\,m} \quad (m > 1) \tag{5.6}$$

Then, if in the general solution (4.8) of the basic nonlinear integral equation (4.1) we restrict ourselves to the first two approximations, using (4.4), (4.5), (4.8) and (5.6), we get

$$\tau_{sz}(t) = \frac{M(\tau_1)}{2h\Omega_*}\Big\{H_0(t, \tau_1) + \beta\left[\frac{M(\tau_1)}{2h\Omega_*}\right]^{m-1}\Big[H_1(t, \tau_1) + \\ + \int_{\tau_1}^{t} H_1(\tau, \tau_1)\, R(t, \tau, \alpha)\, d\tau\Big]\Big\} + O(\beta^2) \tag{5.7}$$

where

$$H_0(t, \tau_1) = 1 - 3\alpha\gamma G\varphi(\tau_1)\frac{e^{r\tau_1}\tau_1^{\,p}}{r^{1-p}}\left[\Phi(rt, p) - \Phi(rt_1, p)\right]$$

$$H_1(t, \tau_1) = \int_{\tau_1}^{t} K(t, \tau)\left[H_0(\tau, \tau_1)\right]^m d\tau \qquad (5.8)$$

$$p = 3\gamma G A_1, \quad r = \gamma\left[1 + 3\alpha G C_0\right], \quad \Phi(\xi, p) = \int_0^{\xi}\frac{e^{-\tau}}{\tau^p}\,d\tau$$

and the incomplete Gamma function has been tabulated.

It should be noted that the first term $H_0(t, \tau_1)$ in (5.7) represents the so-called influence function, characterizing the law of decay in time of the initial elastic stresses in the twisted rod, owing to the influence of linear creep, while the second term of this expression accounts, within a quantity of order β^2, for the influence of the nonlinear creep on the law of decay of these stresses.

Substituting the expression for the stresses $\tau_{sz}(t)$ from (5.7) in (3.5), we find a formula for the determination of the relaxation of the torque: $M(t)$

$$\frac{M(t)}{M(\tau_1)} = H_0(t, \tau_1) + \beta\left[\frac{M(\tau_1)}{2h\Omega_*}\right]^{m-1}\times$$
$$\times\left[H_1(t, \tau_1) + \int_{\tau_1}^{t} R(t, \tau, \alpha) H_1(\tau, \tau_1)\,d\tau\right] + 0(\beta^2) \qquad (5.9)$$

Similarly, if in the general equation (4.8) we limit ourselves to three approximations, for the relaxation of the twisting moment we find the following formula:

$$\frac{M(t)}{M(\tau_1)} = H_0(t, \tau_1) + \beta\left[\frac{M(\tau_1)}{2h\Omega_*}\right]^{m-1}\left[H_1(t, \tau_1) + \int_{\tau_1}^{t} R(t, \tau, \alpha) H_1(\tau, \tau_1)\,d\tau\right] +$$
$$+ m\beta^2\left[\frac{M(\tau_1)}{2h\Omega_*}\right]^{2(m-1)}\left[H_2(t, \tau_1) + \int_{\tau_1}^{t} R(t, \tau, \alpha) H_2(\tau, \tau_1)\,d\tau\right] + 0(\beta^3) \quad (5.10)$$

where

$$H_2(t, \tau_1) = \int_{\tau_1}^{t}\left[H_0(\tau, \tau_1)\right]^{m-1}\left\{H_1(\tau, \tau_1) + \int_{\tau_1}^{\tau} R(t, x, \alpha) H_1(x, \tau_1)\,dx\right\} K(t, \tau)\,d\tau \qquad (5.11)$$

As an application, consider the problem of the relaxation of the torque in a thin-walled concrete rod of box-shaped cross-section (Fig. 2) for the following initial conditions:

$a = 15$ cm	$h = 2$ cm	$\Omega_* = 225$ cm^2
$A_1 = 4.82 \cdot 10^{-5}$	$C_0 = 0.9 \cdot 10^{-5}$	$\gamma = 0.026$
$3G = 2 \cdot 10^5$ kg cm^2	$m = 2$	$M(\tau_1) = 9000$ kg cm^2
	$\tau_1 = 28$ days	

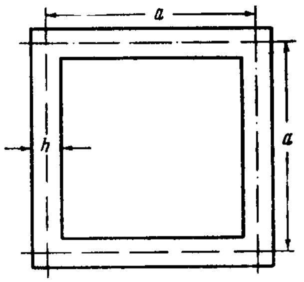

Fig. 2.

For these conditions, consider the four cases corresponding to the following values of the nonlinearity parameter: $\beta = 0$, $\beta = 0.001$, $\beta = 0.01$, $\beta = 0.05$.

Obviously, the value $\beta = 0$ corresponds to the relaxation problem of torsion in the presence of linear creep and it has been considered in [1], while the remaining values of β correspond to relaxation problems for transient creep with quadratic nonlinearity.

The ratio $M(t)/M(\tau_1)$ will be called the decay coefficient of the torque. The value of this coefficient has been tabulated here for different values of time t and the nonlinearity parameter β. It can be seen from this Table that the relaxation process for the torsion of thin-walled rods of closed section in the presence of transient creep is strongly activated by the presence of nonlinearity (for example, for $\beta = 0.5$, the decay coefficient of the torque is two-thirds of that in the case of linear creep).

TABLE 1.

Values of the decay coefficient $M(t)/M(\tau)$

	First approximation	Second approximation			Third approximation		
	$\beta = 0$	$\beta = 0.001$	$\beta = 0.01$	$\beta = 0.05$	$\beta = 0.001$	$\beta = 0.01$	$\beta = 0.05$
45	0.484	0.4838	0.4546	0.3251	0.4838	0.4555	0.3274
90	0.301	0.3001	0.2909	0.2499	0.3001	0.2921	0.2532
180	0.301	0.2999	0.2906	0.2495	0.2999	0.2913	0.2528
360	0.301	0.2998	0.2905	0.2493	0.2998	0.2912	0.2526

It also follows from this Table that for variations of the nonlinearity parameter in the interval $0 < \beta \leqslant 0.01$ (which comprises the range of variation of β in the experimental creep curves of concrete) we may restrict consideration in the general solution (4.8) of the basic equation to two approximations, since we obtain sufficient accuracy thereby.

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Translated by J.R.M.R.